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Localization and Propagation in Random Lattices

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We analyze the motion of a particle on random lattices. Scatterers of two different types are independently distributed among the vertices of such a lattice. A particle hops from a vertex to one of its neighboring vertices. The choice of neighbor is completely determined by the type of scatterer at the current vertex. It is shown that on Poisson and vectorizable random triangular lattices the particle will either propagate along some unbounded strip or be trapped inside a closed strip. We also characterize the structure of a localization zone contained within a closed strip. Another result shows that for a general class of random lattices the orbit of a particle will be bounded with probability one.

KEY WORDS: Lorentz lattice gas; Delaunay lattice; random lattice; localization; propagation; rigid environment.

1. INTRODUCTION

Systems generated by the motion of particles on a lattice, occupied with scatterers randomly distributed over its vertices, have been studied extensively in recent years. The models in which collisions of particles with the scatterers result in changes to an environment (configuration of scatterers), have been especially popular in statistical physics,[(1)] theory of artificial intelligence, theoretical computer science,[(2)] graph theory[(3)] and communication theory.[(4)]

Recently, a class of cellular automata called *walks in rigid environments*, which generalizes all these models, was introduced in ref. 5. According to this paper, an environment is said to have rigidity r if the type of scatterer at a lattice site changes after the r-th visit of a particle to this site.

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It has been shown that models which belong to this class are exactly solvable in dimension one. On a regular triangular lattice the model of rotators constituting a random environment with rigidity one has also been shown to be solvable for the motion of one particle.[6]

In the present paper we study one of these models on random lattices. The natural assumption is made that a particle hops to one of its neighboring vertices at each integer moment of time. We show that on both Poisson and vectorizable Delaunay random lattices (introduced in ref. 7 and ref. 8 respectively), a particle will either propagate in an infinite random strip, or be trapped inside a closed random strip. It is our conjecture that the probability of unbounded orbits is positive. However, the way to prove this result remains unclear. We also demonstrate that the probability of bounded (periodic) orbits is positive. Until now, periodic orbits have been observed numerically for the Poisson lattice[9] but not for the vectorizable lattice.

We also characterize a random environment enclosed inside a periodic orbit. Namely, the average coordination number of vertices in this internal region cannot exceed 6 and is bounded from below by 4.

Finally, we show that the orbit of a particle moving on an arbitrary (not triangular) random lattice is bounded with probability one provided that the lattice satisfies certain natural conditions.

2. LOCALIZATION AND PROPAGATION IN RANDOM TRIANGULAR LATTICES

2.1. Lattice Gases on Delaunay Random Lattices

The Delaunay random lattice[7, 8] is defined as the dual lattice to the Voronoi tessellation of the plane. For a given set of points, the Voronoi tessellation is constructed as follows: for each point of the set we define a cell associated with it as a region of the plane which is nearer to this point than to any other point of the set. Any two such cells sharing an edge are considered neighbors. By drawing a link between every two points associated with neighboring cells we obtain a triangulation of the plane called a Delaunay random lattice. One of the most important properties of this lattice is the following:

Property 2.1. The circle circumscribed around any triangular cell of a Delaunay random lattice does not contain any lattice points inside.

We will consider two known variations of the Delaunay random lattice, a Poisson random lattice (PRL) and a vectorizable random lattice

(VRL), which differ from one another by the initial distribution of the lattice points. The first is obtained by distributing the points randomly and uniformly over the infinite plane. To construct the second lattice we first cover the plane with a regular square lattice and then distribute points randomly and homogeneously inside each square in such a way that each square contains only one point. By connecting the points with links using the procedure described above, we obtain the two variations of the Delaunay random lattice. For further details see e.g. ref. 9.

This paper examines the motion of a particle on both Poisson and vectorizable random lattices. In either case the lattice is fully occupied by rotators that rotate the velocity vector of a particle arriving at a site by the largest possible angle, either to the right (R-rotator) or to the left (L-rotator). We assume that initially the rotators are placed in the vertices of the lattice independently with probabilities P_R and $P_L = 1 - P_R$ respectively. In what follows, all probabilities are computed with respect to this distribution.

The state of a rotator switches to the opposite ($R \Leftrightarrow L$) after each passage of the particle. Therefore this model belongs to the class of *walks in rigid environments* introduced in ref. 5. The corresponding value of the rigidity of environment is one, because only one visit of the particle is required to change the type of scatterer at any site. Following ref. 1 we will refer to this model as the Flipping Rotators (FR) model on a Delaunay random lattice.

In the past, the results of computer simulations based on this model[9] showed the existence of periodic orbits on a PRL but not on a VRL. Here we will investigate the dynamics of the FR model on Delaunay random lattices, of both types, and show that the probability of periodic orbits is positive in both cases.

2.2. Propagation

In this section we will show that the FR model on a Delaunay random lattice exhibits similar behavior as it would on a regular triangular lattice: a moving particle will always propagate in some strip, which we define as follows:

Definition. Given a random lattice **L** we define a strip as a region which is formed by all the sites that a particle visits while performing zigzag motion on **L** (i.e. rotating at each step by the largest possible angle to the left or to the right in alternating order).

Here propagation is defined as it was in the case of a regular lattice (e.g. ref. 6): a particle propagates if its motion is confined to a strip, where each site of the strip can be visited no more than a fixed number of times (unless the strip is bounded).

Proposition 1. For any initial distribution of the scatterers on a Delaunay random lattice, a moving particle always propagates in a strip.

The proof of Proposition 1 repeats that of a similar result for the FR model on a regular triangular lattice.[6] The result is due to the existence of a blocking mechanism that produces a *blocking pattern* based on a zigzag path of length four. On a regular triangular lattice the blocking pattern consists of two pairs of parallel velocity vectors at four consecutive time steps (see Fig. 1). It uniquely defines the propagation strip as well as the direction of propagation. The blocking pattern appearing on a Delaunay random lattice looks like a distorted version of that shown in Fig. 1 due to the irregularity of the lattice. Such a pattern always appears after a finite number of steps (not exceeding 11[6]) and serves two purposes: first, it keeps the orbit of a particle strictly confined to the strip, and second, it prevents the particle from accessing the sites of the strip lying on the opposite side of the blocking pattern. Every time the particle visits a new site of the strip the pattern shifts along the edge of the strip in the direction of propagation thus forcing the particle to continue moving forward. The proof of the result in ref. 6 was based entirely on the triangularity of the lattice and did not require the lattice to be regular. An example of a propagation strip is shown in Fig. 2.

Let us now consider the propagation of a particle in a strip. Every time the particle visits a new site on the strip it will, depending on the state of the scatterer at this site, either a) continue moving onto the next site in the direction of propagation, or b) turn back and make additional 6 steps before visiting the next site along the strip. We will refer to the sites which exhibit outcome a) as *forwarding* sites, and outcome b) as *bouncing* sites. For instance, if a particle propagates from left to right in a horizontal strip then the sites with R-rotators on the "top" boundary of the strip and those with L-rotators on the "bottom" boundary are forwarding sites, whereas the rest are bouncing sites.

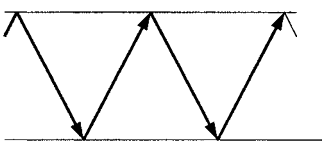

Fig. 1. Blocking pattern appearing on a regular triangular lattice. Arrows indicate the successive velocity vectors of the particle.

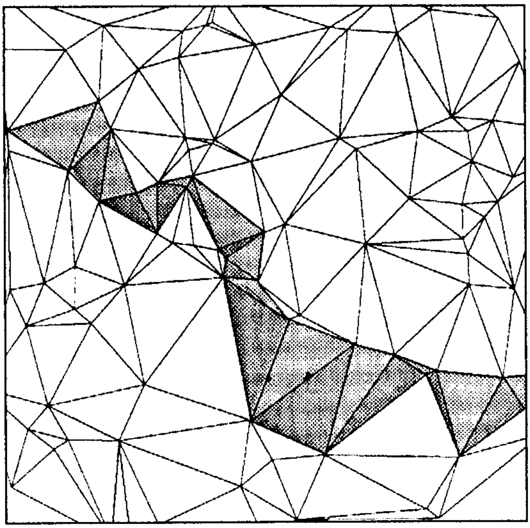

Fig. 2. (Ref. 6, Fig. 1) An example of a propagation strip (shaded area) on a Delaunay random lattice where the particle arriving at a site is deflected over the largest possible angle, either to the right or to the left, depending on the R or L nature of the scatterer. Arrows indicate particle displacements.

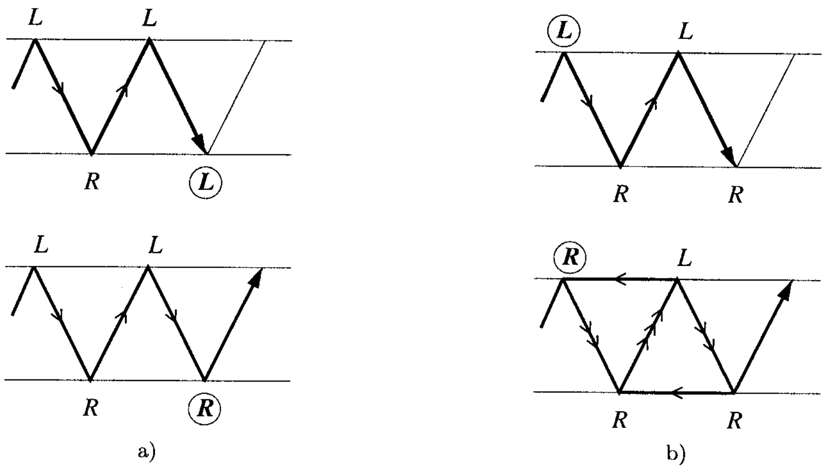

Fig. 3. Reorganization of the medium after the passage of a particle in the case of a) forwarding site; b) bouncing site. Arrows indicate successive velocity vectors of the particle. The number of arrows on each edge corresponds to the number of times the particle traveled along that edge.

When a particle visits a forwarding site, it changes the state of the scatterer at that site to its opposite ($R \Leftrightarrow L$) (see Fig. 3.a)). However, when a particle visits a bouncing site it changes the state of the scatterer at the site situated three steps back along the zigzag path from the bouncing site (see Fig. 3.b)). This observation leads to the following statement, which has recently been formulated in ref. 10:

Proposition 2. The passage of a particle results in a reorganization of the medium in such a way that the initial configuration of the scatterers on the strip shifts three steps back along the zigzag path contained in the strip.

Proof. It follows from our discussion above that site a will become forwarding after the passage of a particle iff site b located three steps ahead of it along the zigzag path, is bouncing. However a and b lie on the opposite boundaries of the strip, therefore a will assume the state $R(L)$ iff b is in the state $R(L)$. Thus, the scatterer at site b gets shifted to site a. Similarly, every scatterer on the strip gets shifted three steps back along the zigzag path. ∎

2.3. Localization

It has been shown in ref. 6 that, on a regular triangular lattice, a particle propagates a strip formed by two parallel lines, and therefore no closed orbits can be found. In Proposition 3 below, we will show that this is not the case for Delaunay random lattices. Before we proceed with the proof however, let us make a few observations:

1) According to Proposition 1, the trajectory of a particle is confined to a strip. Therefore, periodic motion should be confined to some bounded, closed strip, i.e. a strip whose boundaries are closed contours on **L**. In what follows, we will refer to such a strip as a periodic strip. It should also be noted that the inner and outer boundaries of a periodic strip have equal lengths. Hence we can define the length of a periodic strip to be equal to the length of either of its boundaries.

2) A particle will never arrive at a periodic strip nor will it leave it. Therefore, the trajectory of a particle placed at the origin will be periodic only if the origin belongs to some periodic strip.

3) Due to the nature of a Delaunay random lattice, the length of a closed contour on the lattice is bounded from below by 3. We will refer to a strip of length three as a minimal periodic strip, and the orbit confined to this strip as a minimal periodic orbit.

Proposition 3. The probability of closed trajectories for the FR model on a Delaunay random lattice (PRL or VRL) is positive.

2.3.1. *Poisson Random Lattice*

Let **L** be a Poisson random lattice. To prove Proposition 3 we will show that the probability of a minimal periodic trajectory on **L** is positive.

Let C be any triangular cell on **L** with vertices a_1, a_2 and a_3. We will show that, with positive probability, the boundary of this cell is the inner boundary of a minimal strip.

The proof will require additional constructions. Through each vertex a_i, $i = 1, 2, 3$ we will draw a line l_i parallel to the segment $a_{i-1}a_{i+1}$ (the index i is understood in the modulo 3 sense). Since these lines are parallel to the sides of the cell C, they can not be parallel to each other. For each $i = 1, 2, 3$ we will denote the point of intersection of l_{i-1} and l_{i+1} by b_i.

Finally we denote as D_1, D_2 and D_3 the regions bounded by these lines as shown in Fig. 4. Each D_i has a positive measure. Hence with probability one they contain infinitely many lattice points. Notice also that pairwise intersections of D_i's are empty, therefore the distributions of lattice points inside the regions D_i, $i = 1, 2, 3$, are independent.

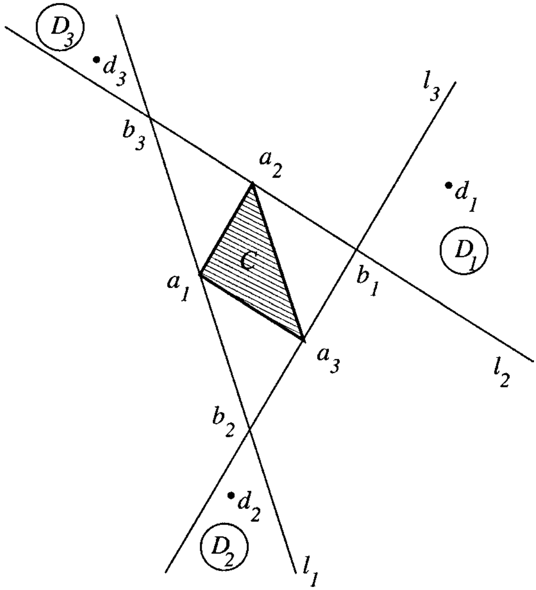

Fig. 4. Construction of minimal periodic strip on PRL in the proof of Proposition 3.

Inside each D_i we choose a lattice point d_i that gives rise to the circle O_i^d, passing through the points d_i, a_{i-1} and a_{i+1}, of the smallest radius. Note that for each i the radius of O_i^d is finite. With probability one each d_i lies in the interior of D_i, therefore the points a_i, d_{i-1} and d_{i+1} do not lie on the same line, and the circle O_i^a passing through them has a finite radius.

Let us recall that Delaunay random lattices satisfy Property 2.1, stated above. Hence the following statements are equivalent:

- The contour $a_1a_2a_3$ is an inner boundary of a periodic strip;
- The coordination number of each site a_i, $i = 1, 2, 3$, is 4;
- The region formed by the union of six circles

$$\left\{\bigcup_{i=1}^{3} O_i^a\right\} \cup \left\{\bigcup_{i=1}^{3} O_i^d\right\}$$

does not contain any lattice points except a_i and d_i, $i = 1, 2, 3$.

The probability of the last event is positive since each of the six circles has a finite radius, and the distribution of the points is uniform over the plane. This proves that the probability that the contour $a_1a_2a_3$ is an inner boundary of the strip is positive.

Since *any* triangular cell with positive probability gives rise to a periodic strip, then so does any of the cells that have a vertex at the origin. Hence, with positive probability, the origin belongs to a periodic strip and a particle placed at the origin has a minimal periodic trajectory.

2.3.2. Vectorizable Random Lattice

Again let $\mathbf{L}$ be a random lattice. Since the construction of a VRL imposes restrictions on the distribution of sites on the plane, the argument we used for the PRL case will not work for a contour on a VRL (it is not clear even whether it is possible to construct a periodic strip of length three on a VRL).

Nevertheless, we can prove that contours, which are the inner boundaries of periodic strips, have positive probability. Let us choose eight disks A_i $i = 1, \ldots, 8$ of a sufficiently small radius r as shown in Fig. 5. Here, the radii of the disks are chosen to be the same for the sake of simplicity. In principle, the radii of A_i can be different. However they must be small enough for the argument below to hold true. If, for each i, we denote the maximum possible radius of A_i as r_i, then $r = min_i r_i$. Next, inside each disk

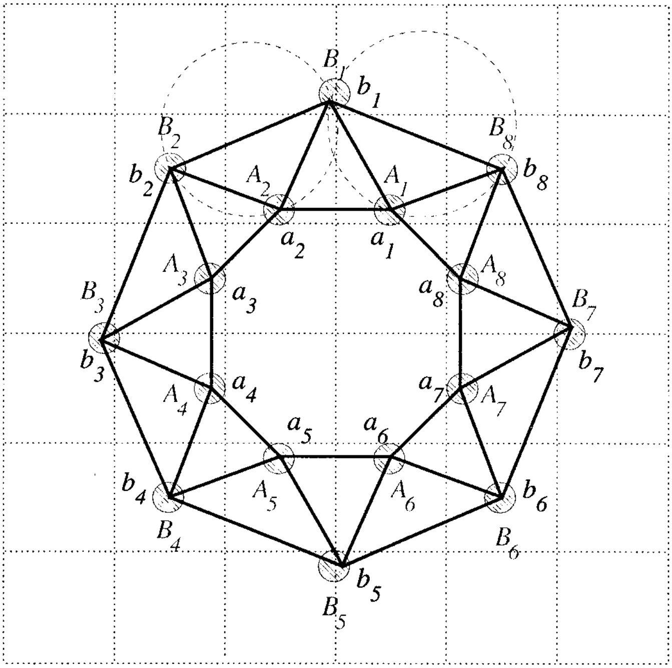

Fig. 5. A closed trajectory of an even length on a VRL. Vertices a_i and b_i, $i = 1, \ldots, 8$, are arbitrary points in the disks A_i and B_i, $i = 1, \ldots, 8$ respectively. Dotted lines indicate the edges of the reference lattice. A circle circumscribed around any cell of the strip must not enclose an "empty" square of the reference lattice. The circles shown indicate the biggest possible circles arising for the chosen set of vertices.

A_i we pick a point a_i and consider a contour $A = a_1 \ldots a_8$. Notice that since the radius r of the disks is nonzero the probability of such a contour on **L** is positive. Now, let us choose eight disks B_i of the same radius r as shown in Fig. 5. Again, the radii are chosen to be the same, and equal to the radius of the disks A_i just for the sake of convenience, but neither of those conditions is essential for the construction.

Claim. Any contour $A = a_1 \ldots a_8$ with $a_i \in A_i$, $i = 1, \ldots, 8$, constructed as above, and a contour $B = b_1 \ldots b_8$ where $b_i \in B_i$, $i = 1, \ldots, 8$ form a strip with positive probability. This strip has A as its inner boundary and B as its outer boundary.

Proof. To prove our Claim let us notice that by Property 2.1, A and B together form a strip iff, for every i, both a circle passing through a_i, b_i and a_{i+1}, and a circle passing through a_i, b_i and b_{i-1} do not contain any other lattice points. This event has a positive probability if the circles do not enclose an entire square of the reference lattice other than one containing a_i or b_i for some i. It is easy to see from Fig. 5 that for any choice of the points a_i and b_i from the disks A_i and B_i respectively all of the mentioned circles satisfy this condition (the two biggest circles arising for our choice of a_i and b_i, $i = 1, \ldots, 8$, are shown in Fig. 5 for verification). This concludes the proof of Claim. ▮

Corollary. Any contour $A = a_1 \ldots a_8$ constructed as above is with positive probability an inner boundary of a periodic strip.

Remark. One can construct longer contours of any even length by adding pairs of segments to the contour in Fig. 5.

The same proof can be repeated for contours of any odd length (see Fig. 6).

Now, combining the Corollary and the fact that the probability of the contour A is positive, we can conclude that the probability of a periodic

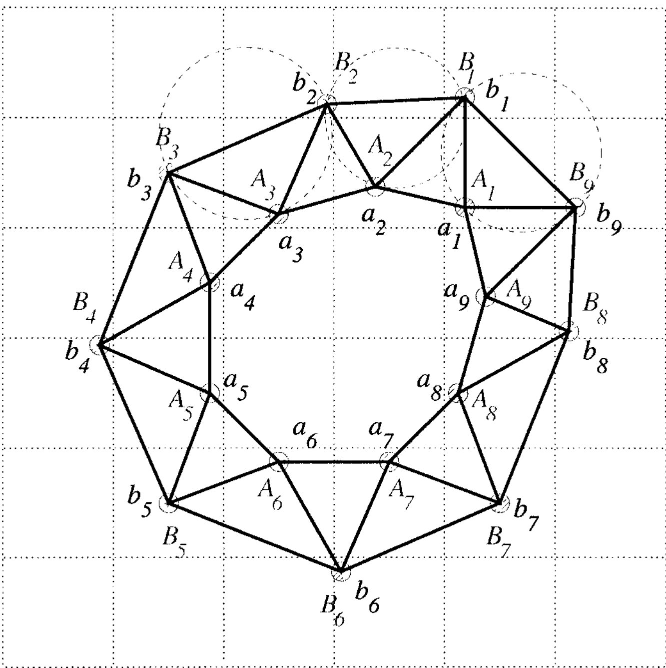

Fig. 6. A closed trajectory of an odd length on a VRL. Here we followed the notations used in Fig. 5.

strip on a VRL is positive. This concludes the proof of Proposition 3 for the case of a VRL.

Remark. Although Proposition 3 demonstrates that the probability of bounded trajectories is positive, it does not answer the question whether it is equal to or strictly less than 1. Our conjecture is that it is strictly less than 1 and, therefore, the probability of unbounded trajectories is positive.

2.4. Remarks on the Periodic Motion

By Proposition 2 the configuration of scatterers shifts along the strip with the passage of a particle. Hence the period of the particle motion in a periodic strip may involve more than one circuit, or passage of the strip. To illustrate this let us choose a periodic strip of length 16 (see Fig. 7), and trace the position of a scatterer originally placed at the origin a_1 with each circuit (see Table I).

We see that during circuits 1, 5 and 9 the scatterer gets shifted twice: first, at the beginning of the circuit, to one of the last three positions on the strip, and then again at the end of the circuit, upon passing those positions. This suggests that for a periodic strip of length n the period of the particle motion may involve as many as $n-3$ circuits.

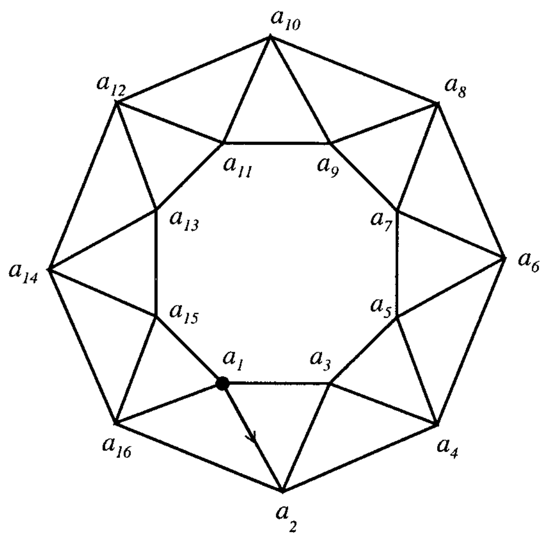

Fig. 7. Periodic strip of length 16 illustrating periodic motion of the particle on a Delaunay random lattice. The dot and the arrow indicate the initial position and velocity vector of the particle. The period of the particle motion may involve as many as 13 circuits.

Table I. Position of a scatterer placed at a_1 after each passage of the periodic strip shown in Fig. 7 by a particle.

Circuit	Position
1	a_{14} then a_{11}
2	a_8
3	a_5
4	a_2
5	a_{15} then a_{12}
6	a_9
7	a_6
8	a_3
9	a_{16} then a_{13}
10	a_{10}
11	a_7
12	a_4
13	a_1

To verify this suggestion, we consider a strip of length n and enumerate the sites of this strip a_1 through a_n in the same way as we did for the previous example. The scatterer undergoes a shift with each passage of a particle. The maximum number of such shifts required for the scatterer to return to its original position is n. However, whenever the scatterer arrives at one of the positions a_1, a_2 or a_3 it gets shifted twice during the next circuit performed by the particle reducing the number of the circuits required by one each time. Hence the maximum number of circuits required to return the configuration to its original state is equal to $n-3$. In the case of $n=16$ the period involves 13 circuits as we see from Table I.

2.5. Necessary Condition for Periodic Orbits

Let us assume that there exists a closed (periodic) orbit. We will find a necessary condition for such an orbit to exist. According to Proposition 1 above this orbit is confined to a closed strip. We will denote the inner boundary of this strip as S_{in} and the outer boundary as S_{out}.

Proposition 4. Let Ω be the region of a random lattice contained inside the boundary S_{in} of a periodic strip including the boundary itself. Then the average coordination number c for this region satisfies the inequality:

$$4 \leqslant c < 6.$$

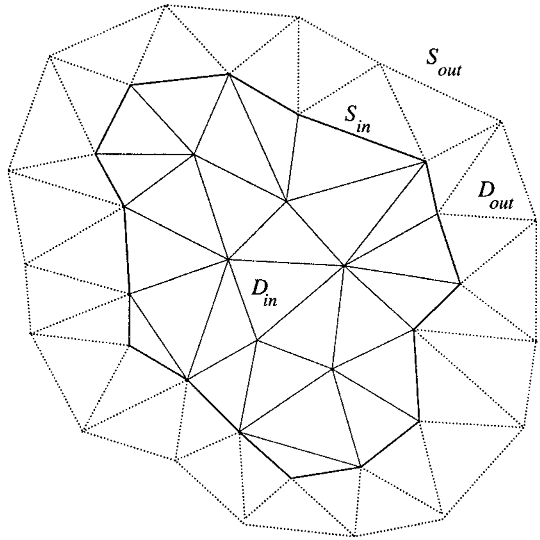

Fig. 8. The region $\Omega = D_{in} \cup S_{in}$ in Proposition 4. Dotted lines indicate the strip enclosing Ω.

Proof. First, we introduce some notations. We denote the region contained strictly inside S_{in} by D_{in}, and the region strictly outside S_{in} by D_{out}. Also let $\Omega = D_{in} \cup S_{in}$ (see Fig. 8) and Ω^c be its complement in $\mathbf{R}^2$.

To compute the average coordination number we find the total number T of links originating at all sites in Ω and divide it by the total number of sites in the region. Let N_S be the number of sites on S_{in} and N_D be the number of sites in D_{in}. It is easy to see that each site of a strip has exactly 4 links in the strip. More precisely, 2 links lie on the boundary and 2 lie inside the strip. Therefore each site of S_{in} has exactly 2 links in D_{out} and 2 links on S_{in} (see Fig. 8).

We split the process of computing T into three steps:

Step 1: the number of links in D_{out}.

Each of the N_S sites of S_{in} has exactly 2 links in D_{out}. Hence the total number of links in D_{out} is equal to $t_1 = 2N_S$.

Step 2: the number of links on S_{in}.

Again each of the N_S boundary sites has exactly 2 links lying on S_{in}. Hence the total number of boundary links is equal to $t_2 = 2N_S$.

Step 3: the number of links in D_{in}.

Let us consider a planar graph G which consists of the triangulated region Ω and an additional face Ω^c. Let t_3 denote the number of edges lying strictly inside D_{in}. Also, let V, F and E be the total number of vertices, faces and edges in the graph G respectively. Then,

$$E = t_3 + N_S, \qquad V = N_S + N_D, \qquad \text{and} \qquad F = \frac{2t_3 + N_S}{3} + 1.$$

Substitution of these values into Euler's formula $V - E + F = 2$ allows us to compute t_3, i.e.:

$$t_3 = N_S + 3(N_D - 1).$$

Now, the total number of links is given by

$$T = t_1 + t_2 + 2t_3 = 4N_S + 2N_S + 6(N_D - 1) = 6(N_S + N_D - 1)$$

To compute the average coordination number we divide it by the total number of sites $S = N_S + N_D$

$$c = \frac{6(N_S + N_D - 1)}{N_S + N_D} = 6\left(1 - \frac{1}{S}\right).$$

As a corollary to the formula above we obtain that $c < 6$ for any number S. Moreover c is an increasing function of S. Therefore, its minimum is attained when S assumes the least possible value, which is 3 (for the case of a minimal periodic strip where $N_D = 0$, $N_S = 3$), and hence is equal to 4. This concludes the proof of Proposition 4. ∎

3. ROTATORS WITH RIGIDITY ONE ON GENERAL RANDOM LATTICES

In this section, we will investigate the dynamics of the Flipping Rotators model on random lattices of a more general type. We consider a lattice satisfying the following two general conditions:

Property 3.1. The configuration shown in Fig. 9, i.e. a triangular cell, with each of its vertices sharing a link with exactly three other sites of the lattice, appears infinitely many times on the lattice.

Property 3.2. With probability one an unbounded trajectory enters any fixed finite configuration infinitely many times.

Property 3.1 seems to be a general property for any random lattice of a generic type where the cells are allowed to be polygons with any number

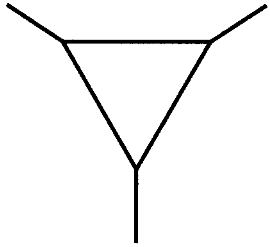

Fig. 9. The triangular cell pattern appearing infinitely many times on random lattices under study (see Property 3.1).

of nodes. Property 3.2 holds if, for instance, a distribution of vertices on a random lattice is translationally invariant.

Proposition 5. In the FR model on a lattice satisfying Properties 3.1 and 3.2 above, the trajectory of a particle is bounded with probability one.

Proof. The proof of Proposition 5 is based on the existence of so called back-scatterers, i.e. clusters formed by sites of the lattice with the following property: if a particle enters the cluster through a certain site, it will leave it after a finite number of steps through the same site with the opposite velocity (see e.g. refs. 5 and 6).

To give an example of such a cluster let us consider the triangular cell C_1 as specified in Property 3.1 and an adjacent cell C_2. Now, let us suppose that all sites of C_2 are occupied by the left rotators L, and the remaining site of C_1 by the right rotator R, as shown in Fig. 10.a). Then it is easy to see that this configuration has a back-scattering property. Indeed, if a particle enters the cluster through the site marked as e, it will first travel counterclockwise along the boundary of C_1, and return back to e. Then it will travel along the boundary of $C_1 \cup C_2$ until it returns to e again. After these two circuits the particle will leave the cluster along the same link it came through originally. Notice that this back-scattering procedure does not bring the configuration of scatterers to its original state, but instead leaves it in the state shown in Fig. 10.b). However, the cluster will return to its original configuration the next time a particle enters this cluster and gets back-scattered. Notice that the probability of such a back-scatterer is positive and equal to $P_L{}^n P_R$ where n is the number of sites in C_2.

To prove Proposition 5, let us assume that the trajectory of a particle is unbounded. Then, by Properties 3.1 and 3.2 it will enter infinitely many

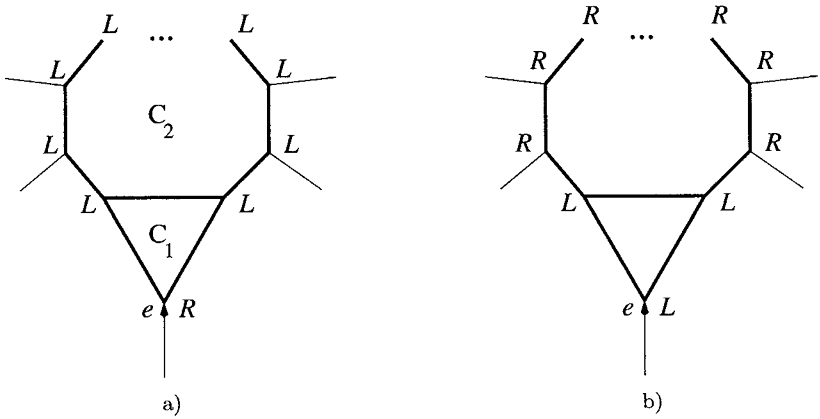

Fig. 10. Two states of a back-scatterer. Each time a particle enters configuration a) (b)) through the site marked as e (the velocity vector of the particle entering the cluster is indicated by the arrow) it gets back-scattered and the configuration of the cluster switches to b) (a)). The next visit of a particle to this cluster will bring it to its initial configuration a) (b)).

triangular cells, and hence infinitely many back-scatterers, although not necessarily through the entrance site e. Since the environment changes after each passage of the particle, we will consider only the first visit of the particle to a cluster. Note that clusters may intersect. Hence, if the particle visits a site that belongs to more than one cluster we will count it as a visit to all of the clusters in the intersection. Under this restriction we can assume the clusters to be independent. By our assumption, the trajectory of the particle is unbounded. Hence, by Property 3.2, it will enter infinitely many independent clusters. However, each cluster consists of finitely many sites. Furthermore, the positions and the orientations of clusters with respect to each other are independent. Therefore, with probability one, the particle will enter some cluster through its entrance point e and be back-scattered.

After back-scattering, due to the time reversibility of the model under study, the particle will retrace its path to the origin and continue its motion. At this point we can apply the same argument to conclude that with probability one it will enter another back-scattering cluster through its entrance point e, and be back-scattered again. The particle will then return to the origin again retracing its path. After that the motion of the particle will become periodic going back and forth between the two back-scattering clusters. This yields a contradiction to our assumption. Hence, the trajectory of a particle is bounded with probability one. ∎

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REFERENCES

1. T. W. Ruigrok and E. G. D.Cohen, *Phys. Lett. A* **133**:435 (1988).
2. C. G. Langton, *Physica D* **22**:120 (1986).
3. A. Gajardo, A. Moreira, and E. Goles, Complexity of Langton Ant, *Disc. App. Math.* (to appear); A. Gajardo and J. Mazoyer, Langton's Ant Evolution in the Degree 3 Graphs, Preprint (1999).
4. Special Issue on general physical systems and the emergence of physical structures from information theory, *Int. J. Gen. Syst.* **27**:1 (1998).
5. L. A .Bunimovich, *Physica A* **279**:169 (2000).
6. P. Grosfils, J. P. Boon, E. G. D. Cohen, and L. A. Bunimovich, *J. Stat. Phys* **97**:575 (1999).
7. N. H. Christ, R. Friedberg, and T. D. Lee, *Nucl. Phys. B* **202**:89 (1982); R. Friedberg and H.-C. Ren, *Nucl. Phys. B* **235[FS11]**:310 (1984).
8. C. Moukarzel and H. J. Herrmann, *J. Stat. Phys* **68**:911 (1992); C. Moukarzel, *Physica A* **190**:13 (1992).
9. F. Wang and E. G. D. Cohen, *J. Stat. Phys* **84**:233 (1996).
10. J. P. Boon, *J. Stat. Phys.* **102**:355 (2001).